

ON MULTIPLE SOLUTIONS FOR NONLOCAL FRACTIONAL PROBLEMS VIA ∇ -THEOREMS

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ABSTRACT. The aim of this paper is to prove multiplicity of solutions for nonlocal fractional equations modeled by

$$\begin{cases} (-\Delta)^s u - \lambda u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$ is fixed, $(-\Delta)^s$ is the fractional Laplace operator, λ is a real parameter, $\Omega \subset \mathbb{R}^n$, $n > 2s$, is an open bounded set with continuous boundary and nonlinearity f satisfies natural superlinear and subcritical growth assumptions. Precisely, along the paper we prove the existence of at least three non-trivial solutions for this problem in a suitable left neighborhood of any eigenvalue of $(-\Delta)^s$. At this purpose we employ a variational theorem of mixed type (one of the so-called ∇ -theorems).

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1. INTRODUCTION

Classical critical point theorems, like the Mountain Pass Theorem, the Linking Theorem or the Saddle Point Theorem (see [2, 26, 27, 37]), have been extensively used in order to construct non-trivial solutions for nonlocal equations of the type

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

under different growth assumptions on f (see, e.g., [3, 4, 7, 9, 11, 19, 28, 30, 32, 33, 34, 35, 36, 38] and references therein, and [6] for a minimization procedure). Here, $s \in (0, 1)$

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is fixed and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$(1.1) \quad -(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n.$$

The aim of this paper is to focus on the existence of multiple solutions for this kind of problems, in the case when f is a superlinear and subcritical nonlinearity. Precisely, we will study the following problem:

$$(1.2) \quad \begin{cases} \mathcal{L}_K u + \lambda u + f(x, u) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Here $s \in (0, 1)$ is fixed, $n > 2s$, $\Omega \subset \mathbb{R}^n$ is an open bounded set with continuous boundary, and the non-local operator \mathcal{L}_K is defined as

$$(1.3) \quad \mathcal{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y) dy, \quad x \in \mathbb{R}^n,$$

and $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ is a function with the properties that

$$(1.4) \quad mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^2, 1\};$$

$$(1.5) \quad \text{there exists } \theta > 0 \text{ such that } K(x) \geq \theta|x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}.$$

A model for K is given by $K(x) = |x|^{-(n+2s)}$. In this case \mathcal{L}_K is the fractional Laplace operator $-(-\Delta)^s$.

In the recent papers [10, 17, 16, 29] the multiplicity of solutions in the nonlocal fractional setting has been addressed by means of classical critical point theorems in the spirit of the ones cited above.

In [14] (see also [13, 15]) Marino and Saccon introduced new critical point theorems, the so-called *theorems of mixed type*, or ∇ -*theorems*, which allow to get multiplicity results for semilinear elliptic problems (see, for instance, [12, 21, 22, 23, 24, 25, 39, 40, 41]).

We think that a natural question is whether or not these techniques may be adapted to the fractional Laplacian setting. One can define a fractional power of the Laplacian using its spectral decomposition: indeed, in [24] the same problem considered along this paper, but for this spectral fractional Laplacian, has been considered. As in [24], the purpose of this paper is to investigate the existence of multiple weak solutions for (1.2).

A weak solution of (1.2) is a solution of the following problem:

$$(1.6) \quad \begin{cases} \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y) dx dy - \lambda \int_{\Omega} u(x) \varphi(x) dx \\ \quad = \int_{\Omega} f(x, u(x)) \varphi(x) dx \quad \forall \varphi \in X_0 \\ u \in X_0 \end{cases}$$

(for this see [31, Lemma 5.6] and [34, footnote 3]). Here X_0 is defined as follows: X is the linear space of Lebesgue measurable functions from \mathbb{R}^n to \mathbb{R} such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

the map $(x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}$ is in $L^2(\mathbb{R}^n \times \mathbb{R}^n \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dx dy)$,

where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

As we said here above, we suppose that equation (1.2) is superlinear and subcritical, that is its right-hand side $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ verifies the following conditions:

$$(1.7) \quad f \text{ is a Carathéodory function;}$$

(1.8) there exist $a_1, a_2 > 0$ and $q \in (2, 2^*)$, $2^* = 2n/(n - 2s)$, such that

$$|f(x, t)| \leq a_1 + a_2 |t|^{q-1} \text{ a.e. } x \in \Omega, t \in \mathbb{R};$$

(1.9) there exist two positive constants a_3 and a_4 such that

$$F(x, t) \geq a_3 |t|^q - a_4 \text{ a.e. } x \in \Omega, t \in \mathbb{R};$$

(1.10) $\lim_{|t| \rightarrow 0} \frac{f(x, t)}{|t|} = 0$ uniformly in $x \in \Omega$;

(1.11) $0 < qF(x, t) \leq tf(x, t)$ a.e. $x \in \Omega, t \in \mathbb{R} \setminus \{0\}$,

where q is given in (1.8) and the function F is the primitive of f with respect to the second variable, that is

(1.12)
$$F(x, t) := \int_0^t f(x, \tau) d\tau.$$

As a model for f we can take the function $f(x, t) = a(x)|t|^{q-2}t$, with $a \in L^\infty(\Omega)$, $\inf_\Omega a > 0$ and $q \in (2, 2^*)$. The exponent 2^* here plays the role of a fractional critical Sobolev exponent (see, e.g. [8, Theorem 6.5]).

Remark 1. As remarked in [20], condition (1.9) is not a mere consequence of (1.11), and must be assumed *a priori*, unless $f : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and (1.8) holds for every $(x, t) \in \bar{\Omega} \times \mathbb{R}$.

The main result of the present paper can be stated as follows:

Theorem 2. *Let $s \in (0, 1)$, $n > 2s$ and Ω be an open bounded subset of \mathbb{R}^n with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ be a function satisfying (1.4) and (1.5) and let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function verifying (1.7)–(1.11).*

Then, for every eigenvalue λ_k of $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, there exists a left neighborhood \mathcal{O}_k of λ_k such that problem (1.2) admits at least three non-trivial weak solutions for all $\lambda \in \mathcal{O}_k$.

In the non-local framework, the simplest example we can deal with is given by the fractional Laplacian, according to the following result:

Theorem 3. *Let $s \in (0, 1)$, $n > 2s$ and Ω be an open bounded subset of \mathbb{R}^n with continuous boundary. If $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a function verifying (1.7)–(1.11), then the problem*

(1.13)
$$\begin{cases} (-\Delta)^s u - \lambda u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

admits at least three non-trivial weak solutions belonging to $H^s(\mathbb{R}^n)$ in a suitable left neighborhood of any eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data.

When $s = 1$, problem (1.13) reduces to a standard semilinear Laplace equation: in this sense Theorem 3 may be seen as the fractional version of the result in [22, Theorem 1].

We prove Theorem 2 employing variational and topological methods. Precisely, we apply a ∇ -theorem due to Marino and Saccon in [14], see Theorem 9 below. The main difficulties in applying such a theorem are obviously related to the nonlocal nature of the problem.

The paper is organized as follows. In Section 2 we collect the notation and some preliminary observations. In Section 3 we discuss the compactness property of the Euler-Lagrange functional associated with the problem under consideration, while Section 4 is devoted to its geometric structure. In Section 5 we prove the ∇ -condition, which is one of the main ingredient of the critical point theorem we employ in order to get our multiplicity result. Finally, in Section 6 we prove Theorem 2.

2. PRELIMINARIES

In this section we give some preliminary results.

2.1. Variational setting. First of all, we need some notations. In the sequel we endow the space X_0 with the norm defined as (see [32, Lemma 6])

$$(2.1) \quad \|g\|_{X_0} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x) - g(y)|^2 K(x - y) dx dy \right)^{1/2},$$

which is obviously related to the so-called *Gagliardo norm*

$$(2.2) \quad \|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}$$

of the usual fractional Sobolev space $H^s(\Omega)$. For further details on the fractional Sobolev spaces we refer to [1, 8, 18] and to the references therein.

Problem (1.6) has a variational structure: indeed, it is the Euler-Lagrange equation of the functional $\mathcal{J}_\lambda : X_0 \rightarrow \mathbb{R}$ defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy - \frac{\lambda}{2} \int_{\Omega} |u(x)|^2 dx - \int_{\Omega} F(x, u(x)) dx.$$

Note that when functional \mathcal{J}_λ is Fréchet differentiable at $u \in X_0$, we have that for any $\varphi \in X_0$

$$\begin{aligned} \langle \mathcal{J}'_\lambda(u), \varphi \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x - y) dx dy - \lambda \int_{\Omega} u(x) \varphi(x) dx \\ &\quad - \int_{\Omega} f(x, u(x)) \varphi(x) dx, \end{aligned}$$

where we have denoted by $\langle \cdot, \cdot \rangle$ the duality between X'_0 and X_0 . Thus, critical points of \mathcal{J}_λ are solutions to problem (1.6). In order to find multiplicity of critical points, we will use the ∇ -theorem in the form of Theorem 9 (see Section 6).

2.2. Estimates on the nonlinearity. Here, we recall some estimates on the nonlinear term and its primitive, which will be useful in the sequel. These estimates are quite standard and do not take into account the non-local features of the problem. For a proof we refer to [32, Lemma 3 and Lemma 4].

By assumptions (1.7), (1.8) and (1.10) we deduce that

$$(2.3) \quad \text{for any } \varepsilon > 0 \text{ there exists } C_\varepsilon > 0 \text{ such that} \\ |f(x, t)| \leq 2\varepsilon |t| + qC_\varepsilon |t|^{q-1} \text{ a.e. } x \in \Omega, t \in \mathbb{R}$$

and so, as a consequence,

$$(2.4) \quad |F(x, t)| \leq \varepsilon |t|^2 + C_\varepsilon |t|^q,$$

where F is defined as in (1.12). This implies that functional \mathcal{J}_λ is Fréchet differentiable at any point $u \in X_0$.

2.3. An eigenvalue problem for $-\mathcal{L}_K$. Along the present paper we consider the following eigenvalue problem associated to the integro-differential operator $-\mathcal{L}_K$:

$$(2.5) \quad \begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

We denote by $\{\lambda_k\}_{k \in \mathbb{N}}$ the sequence of the eigenvalues of the problem (2.5), with

$$(2.6) \quad 0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \lambda_{k+1} \leq \dots \\ \lambda_k \rightarrow +\infty \text{ as } k \rightarrow +\infty,$$

and by e_k the eigenfunction corresponding to λ_k . Moreover, we normalize $\{e_k\}_{k \in \mathbb{N}}$ in such a way that this sequence provides an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis

of X_0 . For a complete study of the spectrum of the integro-differential operator $-\mathcal{L}_K$ we refer to [28, Proposition 2.3], [33, Proposition 9 and Appendix A] and [35, Proposition 4]. In particular, we recall that all eigenfunctions are Hölder continuous up to the boundary of Ω .

Finally, we say that eigenvalue λ_k , $k \geq 2$, has multiplicity $m \in \mathbb{N}$ if

$$\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}.$$

In this case the set of all the eigenvalues corresponding to λ_k agrees with

$$\text{span} \{e_k, \dots, e_{k+m-1}\}.$$

In the following, for any $k \in \mathbb{N}$, we set

$$\mathbb{H}_k = \text{span} \{e_1, \dots, e_k\}$$

and

$$\mathbb{P}_k = \left\{ u \in X_0 : \langle u, e_j \rangle_{X_0} = 0 \text{ for any } j = 1, \dots, k \right\},$$

where

$$\langle u, v \rangle_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x - y) dx dy$$

makes X_0 a Hilbert space, see [32, Lemma 7]. In this way, the variational characterization of the eigenvalues (see [33, Proposition 9]) implies that

$$(2.7) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \geq \lambda_{k+1} \int_{\Omega} |u(x)|^2 dx \text{ for all } u \in \mathbb{P}_k,$$

while, using the orthogonality properties of the eigenvalues, a standard Fourier decomposition gives

$$(2.8) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \leq \lambda_k \int_{\Omega} |u(x)|^2 dx \text{ for all } u \in \mathbb{H}_k.$$

2.4. Gradient in X_0 . Being X_0 a Hilbert space and \mathcal{J}_λ of class C^1 , the gradient $\nabla \mathcal{J}_\lambda$ of \mathcal{J}_λ is immediately defined as

$$(2.9) \quad \begin{aligned} \langle \nabla \mathcal{J}_\lambda(u), v \rangle_{X_0} &:= \langle \mathcal{J}'_\lambda(u), v \rangle \\ &= \langle u, v \rangle_{X_0} - \lambda \int_{\Omega} u(x)v(x) dx - \int_{\Omega} f(x, u(x))v(x) dx \end{aligned}$$

for any $u, v \in X_0$.

Let $\nu \in [1, 2^*]$ and ν' be its conjugate, that is $1/\nu + 1/\nu' = 1$. Introducing the operator $\mathcal{L}_K^{-1} : L^{\nu'}(\Omega) \rightarrow X_0$, defined as $\mathcal{L}_K^{-1}g = v$ if and only if $v \in X_0$ solves

$$\begin{cases} \mathcal{L}_K v = g(x) & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

it is readily seen that

$$(2.10) \quad \nabla \mathcal{J}_\lambda(u) = u - \mathcal{L}_K^{-1}(\lambda u + f(x, u))$$

for all $u \in X_0$. Indeed, setting $w = \mathcal{L}_K^{-1}(\lambda u + f(x, u))$ and using the definitions of \mathcal{L}_K and \mathcal{J}_λ , for any test function $\varphi \in X_0$ we get that

$$\begin{aligned} \langle w, \varphi \rangle_{X_0} &= \lambda \int_{\Omega} u(x)\varphi(x) dx + \int_{\Omega} f(x, u(x))\varphi(x) dx \\ &= -\langle \mathcal{J}'_\lambda(u), \varphi \rangle + \langle u, \varphi \rangle_{X_0} \\ &= -\langle \nabla \mathcal{J}_\lambda(u), v \rangle_{X_0} + \langle u, \varphi \rangle_{X_0}, \end{aligned}$$

that is

$$w = -\nabla \mathcal{J}_\lambda(u) + u,$$

which gives the assertion.

Moreover, we note that \mathcal{L}_K^{-1} is a compact operator for all $\nu \in [1, 2^*)$. Indeed, if $\{g_n\}_{n \in \mathbb{N}}$ is bounded in $L^{\nu'}(\Omega)$, standard calculations imply that $\{\mathcal{L}_K^{-1}g_n\}_{n \in \mathbb{N}}$ is relatively compact in X_0 .

For further use, we also note that, again using the definition of \mathcal{L}_K^{-1} ,

$$(2.11) \quad \langle u, \mathcal{L}_K^{-1}v \rangle_{X_0} = \int_{\Omega} u(x)v(x) dx$$

for every $u, v \in X_0$.

3. COMPACTNESS CONDITION

In this section we check the validity of the *Palais–Smale condition* for functional \mathcal{J}_λ at any level, that is we prove that for each $c \in \mathbb{R}$ any Palais–Smale sequence for \mathcal{J}_λ at level c admits a subsequence which is strongly convergent in X_0 . As usual, we say that $\{u_j\}_{j \in \mathbb{N}} \subset X_0$ is a *Palais–Smale sequence* for \mathcal{J}_λ at level $c \in \mathbb{R}$ if

$$(3.1) \quad \mathcal{J}_\lambda(u_j) \rightarrow c$$

and

$$(3.2) \quad \sup \left\{ |\langle \mathcal{J}'_\lambda(u_j), \varphi \rangle| : \varphi \in X_0, \|\varphi\|_{X_0} = 1 \right\} \rightarrow 0$$

as $j \rightarrow +\infty$.

Proposition 4. *Let $\lambda > 0$ and f be a function satisfying conditions (1.7)–(1.11).*

Then, functional \mathcal{J}_λ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j \in \mathbb{N}}$ be a Palais–Smale sequence for \mathcal{J}_λ at level c . First of all, let us prove that

$$(3.3) \quad \text{the sequence } \{u_j\}_{j \in \mathbb{N}} \text{ is bounded in } X_0.$$

For this purpose, we note that by (3.1) and (3.2) for any $j \in \mathbb{N}$ it easily follows that there exists $\kappa > 0$ such that

$$\left| \langle \mathcal{J}'_\lambda(u_j), \frac{u_j}{\|u_j\|_{X_0}} \rangle \right| \leq \kappa,$$

and

$$|\mathcal{J}_\lambda(u_j)| \leq \kappa,$$

so that, as a consequence of these two relations, we also have

$$(3.4) \quad \mathcal{J}_\lambda(u_j) - \frac{1}{\mu} \langle \mathcal{J}'_\lambda(u_j), u_j \rangle \leq \kappa (1 + \|u_j\|_{X_0}),$$

where μ is a parameter such that $\mu \in (2, q)$.

Now, thanks to (1.11) and (1.9) we get

$$(3.5) \quad \begin{aligned} \mathcal{J}_\lambda(u_j) - \frac{1}{\mu} \langle \mathcal{J}'_\lambda(u_j), u_j \rangle &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(\|u_j\|_{X_0}^2 - \lambda \|u_j\|_{L^2(\Omega)}^2 \right) \\ &\quad - \int_{\Omega} \left(F(x, u_j(x)) - \frac{1}{\mu} f(x, u_j(x)) u_j(x) \right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(\|u_j\|_{X_0}^2 - \lambda \|u_j\|_{L^2(\Omega)}^2 \right) \\ &\quad + \left(\frac{q}{\mu} - 1 \right) \int_{\Omega} F(x, u_j(x)) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \left(\|u_j\|_{X_0}^2 - \lambda \|u_j\|_{L^2(\Omega)}^2 \right) \\ &\quad + a_3 \left(\frac{q}{\mu} - 1 \right) \|u_j\|_{L^q(\Omega)}^q - a_4 \left(\frac{q}{\mu} - 1 \right) |\Omega|. \end{aligned}$$

Note that for any $\varepsilon > 0$ the Young inequality gives

$$(3.6) \quad \|u_j\|_{L^2(\Omega)}^2 \leq \frac{2\varepsilon}{q} \|u_j\|_{L^q(\Omega)}^q + \frac{q-2}{q} \varepsilon^{-2/(q-2)} |\Omega|,$$

so that, by (3.5) and (3.6), we obtain that

$$(3.7) \quad \begin{aligned} \mathcal{J}_\lambda(u_j) - \frac{1}{\mu} \langle \mathcal{J}'_\lambda(u_j), u_j \rangle &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_j\|_{X_0}^2 - \lambda \left(\frac{1}{2} - \frac{1}{\mu} \right) \frac{2\varepsilon}{q} \|u_j\|_{L^q(\Omega)}^q \\ &\quad - \lambda \left(\frac{1}{2} - \frac{1}{\mu} \right) \frac{q-2}{q} \varepsilon^{-2/(q-2)} |\Omega| \\ &\quad + a_3 \left(\frac{q}{\mu} - 1 \right) \|u_j\|_{L^q(\Omega)}^q - a_4 \left(\frac{q}{\mu} - 1 \right) |\Omega| \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_j\|_{X_0}^2 \\ &\quad + \left[a_3 \left(\frac{q}{\mu} - 1 \right) - \lambda \left(\frac{1}{2} - \frac{1}{\mu} \right) \frac{2\varepsilon}{q} \right] \|u_j\|_{L^q(\Omega)}^q - C_\varepsilon, \end{aligned}$$

where C_ε is a constant such that $C_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, being $q > \mu > 2$.

Now, choosing ε so small that

$$a_3 \left(\frac{q}{\mu} - 1 \right) - \lambda \left(\frac{1}{2} - \frac{1}{\mu} \right) \frac{2\varepsilon}{q} > 0,$$

by (3.7) we get

$$(3.8) \quad \mathcal{J}_\lambda(u_j) - \frac{1}{\mu} \langle \mathcal{J}'_\lambda(u_j), u_j \rangle \geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_j\|_{X_0}^2 - C_\varepsilon.$$

Finally, by (3.4) and (3.8) for any $j \in \mathbb{N}$

$$\|u_j\|_{X_0}^2 \leq \kappa_* (1 + \|u_j\|_{X_0})$$

for a suitable positive constant κ_* . Hence, assertion (3.3) is proved.

Now, let us finish the proof of the Palais–Smale condition for \mathcal{J}_λ . Since $\{u_j\}_{j \in \mathbb{N}}$ is bounded in X_0 and X_0 is a reflexive space, up to a subsequence, still denoted by u_j , there exists $u_\infty \in X_0$ such that

$$(3.9) \quad \begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y)) (\varphi(x) - \varphi(y)) K(x - y) dx dy &\rightarrow \\ \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_\infty(x) - u_\infty(y)) (\varphi(x) - \varphi(y)) K(x - y) dx dy &\quad \text{for any } \varphi \in X_0, \end{aligned}$$

while, by [32, Lemma 8], up to a subsequence,

$$(3.10) \quad \begin{aligned} u_j &\rightarrow u_\infty \quad \text{in } L^\nu(\mathbb{R}^n) \quad \text{for any } \nu \in [1, 2^*) \\ u_j &\rightarrow u_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned}$$

as $j \rightarrow +\infty$. Finally, by [5, Theorem IV.9] we know that for any $\nu \in [1, 2^*)$ there exists $\ell_\nu \in L^\nu(\mathbb{R}^n)$ such that

$$(3.11) \quad |u_j(x)| \leq \ell_\nu(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}.$$

By (1.8), (3.10), (3.11), the fact that the map $t \mapsto f(\cdot, t)$ is continuous in $t \in \mathbb{R}$ (see (1.7)) and the Dominated Convergence Theorem, we get

$$(3.12) \quad \int_{\Omega} f(x, u_j(x)) u_j(x) dx \rightarrow \int_{\Omega} f(x, u_\infty(x)) u_\infty(x) dx$$

and

$$(3.13) \quad \int_{\Omega} f(x, u_j(x)) u_\infty(x) dx \rightarrow \int_{\Omega} f(x, u_\infty(x)) u_\infty(x) dx$$

as $j \rightarrow +\infty$. Furthermore, by (3.2) and (3.3) we have that

$$\begin{aligned} 0 \leftarrow \langle \mathcal{J}'_\lambda(u_j), u_j \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y) dx dy - \lambda \int_{\Omega} |u_j(x)|^2 dx \\ &\quad - \int_{\Omega} f(x, u_j(x)) u_j(x) dx \end{aligned}$$

so that, by (3.10) and (3.12) we deduce that

$$(3.14) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y) dx dy \rightarrow \lambda \int_{\Omega} |u_\infty(x)|^2 dx + \int_{\Omega} f(x, u_\infty(x)) u_\infty(x) dx$$

as $j \rightarrow +\infty$, while, by (3.2), (3.9) (both with test function $\varphi = u_\infty$), (3.10) and (3.13), we get

$$(3.15) \quad \begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_\infty(x) - u_\infty(y)|^2 K(x-y) dx dy &= \lambda \int_{\Omega} |u_\infty(x)|^2 dx \\ &\quad + \int_{\Omega} f(x, u_\infty(x)) u_\infty(x) dx. \end{aligned}$$

Hence, (3.14) and (3.15) give that

$$(3.16) \quad \|u_j\|_{X_0} \rightarrow \|u_\infty\|_{X_0}$$

as $j \rightarrow +\infty$. With this, it is easy to see that

$$\begin{aligned} \|u_j - u_\infty\|_{X_0}^2 &= \|u_j\|_{X_0}^2 + \|u_\infty\|_{X_0}^2 - 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x-y) dx dy \\ &\rightarrow 2\|u_\infty\|_{X_0}^2 - 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_\infty(x) - u_\infty(y)|^2 K(x-y) dx dy = 0 \end{aligned}$$

as $j \rightarrow +\infty$, thanks to (3.9) and (3.16). Then, the proof of Proposition 4 is complete. \square

4. GEOMETRY OF THE ∇ -THEOREM

In this section we check that functional \mathcal{J}_λ has the geometric structure required by the ∇ -theorem stated in Theorem 9 (see Section 6). Precisely, we want to show that if there exist k and m in \mathbb{N} such that

$$\lambda_{k-1} < \lambda < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$$

and λ is sufficiently close to λ_k , then functional \mathcal{J}_λ agrees with the geometric framework of Theorem 9, taking

$$\begin{aligned} X_1 &:= \mathbb{H}_{k-1} \\ X_2 &:= \text{span}\{e_k, \dots, e_{k+m-1}\} \\ X_3 &:= \mathbb{P}_{k+m-1}. \end{aligned}$$

Proposition 5. *Let k and m in \mathbb{N} be such that $\lambda_{k-1} < \lambda < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$ and let f be a function satisfying conditions (1.7)–(1.11).*

Then, there exist ρ, R , with $R > \rho > 0$, such that

$$\sup_{\{u \in X_1, \|u\|_{X_0} \leq R\} \cup \{u \in X_1 \oplus X_2 : \|u\| = R\}} \mathcal{J}_\lambda(u) < \inf_{\{u \in X_2 \oplus X_3 : \|u\|_{X_0} = \rho\}} \mathcal{J}_\lambda(u).$$

Proof. First of all, let us show that

$$(4.1) \quad \inf_{\{u \in X_2 \oplus X_3 : \|u\|_{X_0} = \rho\}} \mathcal{J}_\lambda(u) > 0.$$

For this purpose, let u be a function in $X_2 \oplus X_3 = \mathbb{P}_{k-1}$. By (2.4), we get that for any $\varepsilon > 0$

$$(4.2) \quad \begin{aligned} \mathcal{J}_\lambda(u) &\geq \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy - \frac{\lambda}{2} \int_{\Omega} |u(x)|^2 dx \\ &\quad - \varepsilon \int_{\Omega} |u(x)|^2 dx - C_\varepsilon \int_{\Omega} |u(x)|^q dx. \end{aligned}$$

Moreover, by (2.7), we get that for any $u \in \mathbb{P}_{k-1}$

$$\lambda_k \int_{\Omega} |u(x)|^2 dx \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy,$$

so that this inequality and (4.2) give

$$\begin{aligned} \mathcal{J}_{\lambda}(u) &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_k}\right) \|u\|_{X_0}^2 - \varepsilon \|u\|_{L^2(\Omega)}^2 - C_{\varepsilon} \|u\|_{L^q(\Omega)}^q \\ (4.3) \quad &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_k}\right) \|u\|_{X_0}^2 - \varepsilon |\Omega|^{(2^*-2)/2^*} \|u\|_{L^{2^*}(\Omega)}^2 - |\Omega|^{(2^*-q)/2^*} C_{\varepsilon} \|u\|_{L^{2^*}(\Omega)}^q. \end{aligned}$$

Here we used also the fact that $L^{2^*}(\Omega) \hookrightarrow L^{\nu}(\Omega)$ continuously, being Ω bounded and $\nu \in [2, 2^*)$ (here ν takes the values 2 and q).

Using (1.5), (4.3) and [32, Lemma 6], we obtain that for any $\varepsilon > 0$,

$$\begin{aligned} \mathcal{J}_{\lambda}(u) &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_k}\right) \|u\|_{X_0}^2 - \varepsilon c |\Omega|^{(2^*-2)/2^*} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \\ &\quad - C_{\varepsilon} c^{q/2} |\Omega|^{(2^*-q)/2^*} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{q/2} \\ (4.4) \quad &\geq \left[\frac{1}{2} \left(1 - \frac{\lambda}{\lambda_k}\right) - \frac{\varepsilon c |\Omega|^{(2^*-2)/2^*}}{\theta} \right] \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \\ &\quad - \frac{C_{\varepsilon} c^{q/2} |\Omega|^{(2^*-q)/2^*}}{\theta} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \right)^{q/2}, \end{aligned}$$

where c is a suitable universal positive constant.

Choosing $\varepsilon > 0$ such that

$$2\varepsilon c |\Omega|^{(2^*-2)/2^*} < \theta \left(1 - \frac{\lambda}{\lambda_k}\right),$$

inequality (4.4) reads as

$$\mathcal{J}_{\lambda}(u) \geq \alpha \|u\|_{X_0}^2 \left(1 - \kappa \|u\|_{X_0}^{q-2}\right),$$

for suitable positive constants α and κ .

Now, let $\rho > 0$ be sufficiently small, i.e. ρ such that $1 - \kappa \rho^{q-2} > 0$. Then, for any $u \in \mathbb{P}_{k-1}$ such that $\|u\|_{X_0} = \rho$ we get that

$$\mathcal{J}_{\lambda}(u) \geq \alpha \rho^2 (1 - \kappa \rho^{q-2}) > 0,$$

so that (4.1) is proved.

Now, let us show that

$$(4.5) \quad \sup_{\{u \in X_1, \|u\|_{X_0} \leq R\} \cup \{u \in X_1 \oplus X_2 : \|u\| = R\}} \mathcal{J}_{\lambda}(u) \leq 0.$$

First of all, let us take $u \in \mathbb{H}_{k-1}$. Then

$$u(x) = \sum_{i=1}^{k-1} u_i e_i(x),$$

with $u_i \in \mathbb{R}$, $i = 1, \dots, k-1$ and so, by (2.8) and (1.11), we deduce that

$$(4.6) \quad \mathcal{J}_{\lambda}(u) \leq \frac{\lambda_{k-1} - \lambda}{2} \int_{\Omega} |u(x)|^2 dx \leq 0,$$

since $\lambda_{k-1} < \lambda$.

Finally, let us consider $u \in X_1 \oplus X_2 = \mathbb{H}_k$. By (1.9) we have

$$\mathcal{J}_{\lambda}(u) \leq \frac{1}{2} \|u\|_{X_0}^2 - a_3 \|u\|_{L^q(\Omega)}^q + a_4 |\Omega|,$$

and the claim follows recalling that $q > 2$ and that \mathbb{H}_k is a finite-dimensional subspace of X_0 . This and (4.6) give (4.5).

Then, the assertion of Proposition 5 comes trivially from (4.1) and (4.5). \square

5. ∇ -CONDITION

One of the main ingredient of the ∇ -theorem (see [14, Theorem 2.10]) we employ in order to get our multiplicity result is the so-called ∇ -condition introduced in [14, Definition 2.4]. This section is devoted to the verification of this condition for functional \mathcal{J}_λ . For this purpose, in the sequel we denote by

$$P_C : X_0 \rightarrow C$$

the orthogonal projection of X_0 onto C .

Let C be a closed subspace of X_0 and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that functional \mathcal{J}_λ verifies condition $(\nabla)(\mathcal{J}_\lambda, C, a, b)$ if there exists $\gamma > 0$ such that

$$\inf \left\{ \|P_C \nabla \mathcal{J}_\lambda(u)\|_{X_0} : a \leq \mathcal{J}_\lambda(u) \leq b, \text{ dist}(u, C) \leq \gamma \right\} > 0.$$

Roughly speaking, the condition $(\nabla)(\mathcal{J}_\lambda, C, a, b)$ requires that \mathcal{J}_λ has no critical points $u \in C$ such that $a \leq \mathcal{J}_\lambda(u) \leq b$, with some uniformity. In order to prove this condition for \mathcal{J}_λ , we need two preliminary lemmas.

Lemma 6. *Let k and m in \mathbb{N} be such that $\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$ and let f be a function satisfying conditions (1.7)–(1.11).*

Then, for any $\sigma > 0$ there exists $\varepsilon_\sigma > 0$ such that for any $\lambda \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ the unique critical point u of \mathcal{J}_λ constrained on $\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}$ and with $\mathcal{J}_\lambda(u) \in [-\varepsilon_\sigma, \varepsilon_\sigma]$, is the trivial one.

Proof. We argue by contradiction and we suppose that there exists $\bar{\sigma} > 0$, a sequence $\{\mu_j\}_{j \in \mathbb{N}}$ in \mathbb{R} with

$$(5.1) \quad \mu_j \in [\lambda_{k-1} + \bar{\sigma}, \lambda_{k+m} - \bar{\sigma}]$$

and a sequence $\{u_j\}_{j \in \mathbb{N}}$ such that

$$(5.2) \quad u_j \in \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1} \setminus \{0\},$$

$$(5.3) \quad \langle \mathcal{J}'_{\mu_j}(u_j), \varphi \rangle = 0 \quad \text{for any } \varphi \in \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1},$$

for any $j \in \mathbb{N}$, and

$$(5.4) \quad \begin{aligned} \mathcal{J}_{\mu_j}(u_j) &= \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x - y) dx dy - \frac{\mu_j}{2} \int_{\Omega} |u_j(x)|^2 dx \\ &\quad - \int_{\Omega} F(x, u_j(x)) dx \rightarrow 0 \end{aligned}$$

as $j \rightarrow +\infty$.

Taking $\varphi = u_j$ in (5.3) (this is possible thanks to (5.2)) and using (1.11), we get that for any $j \in \mathbb{N}$

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x - y) dx dy - \mu_j \int_{\Omega} |u_j(x)|^2 dx - \int_{\Omega} f(x, u_j(x)) u_j(x) dx \\ &= 2\mathcal{J}_{\mu_j}(u_j) + \int_{\Omega} \left(2F(x, u_j(x)) - f(x, u_j(x)) u_j(x) \right) dx \\ &\leq 2\mathcal{J}_{\mu_j}(u_j) + (2 - q) \int_{\Omega} F(x, u_j(x)) dx. \end{aligned}$$

Hence, by this inequality, the fact that $q > 2$ and again (1.11), we deduce that

$$0 < (q - 2) \int_{\Omega} F(x, u_j(x)) dx \leq 2\mathcal{J}_{\mu_j}(u_j) \rightarrow 0$$

thanks to (5.4), so that we get

$$(5.5) \quad \int_{\Omega} F(x, u_j(x)) dx \rightarrow 0$$

as $j \rightarrow +\infty$.

Now, since (5.2) holds true, for any $j \in \mathbb{N}$ there exist $v_j \in \mathbb{H}_{k-1}$ and $w_j \in \mathbb{P}_{k+m-1}$ such that

$$u_j = v_j + w_j.$$

Letting $\varphi = v_j - w_j$ in (5.3) and taking into account the orthogonality properties of v_j and w_j , we have that for any $j \in \mathbb{N}$

$$(5.6) \quad \begin{aligned} 0 &= \langle \mathcal{J}'_{\mu_j}(u_j), v_j - w_j \rangle \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |v_j(x) - v_j(y)|^2 K(x - y) dx dy - \int_{\mathbb{R}^n \times \mathbb{R}^n} |w_j(x) - w_j(y)|^2 K(x - y) dx dy \\ &\quad - \mu_j \int_{\Omega} |v_j(x)|^2 dx + \mu_j \int_{\Omega} |w_j(x)|^2 dx - \int_{\Omega} f(x, u_j(x))(v_j(x) - w_j(x)) dx. \end{aligned}$$

By (2.8) and (2.7), (5.6) gives that for any $j \in \mathbb{N}$

$$(5.7) \quad \begin{aligned} \int_{\Omega} f(x, u_j(x))(v_j(x) - w_j(x)) dx &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |v_j(x) - v_j(y)|^2 K(x - y) dx dy \\ &\quad - \int_{\mathbb{R}^n \times \mathbb{R}^n} |w_j(x) - w_j(y)|^2 K(x - y) dx dy \\ &\quad - \mu_j \int_{\Omega} |v_j(x)|^2 dx + \mu_j \int_{\Omega} |w_j(x)|^2 dx \\ &\leq \frac{\lambda_{k-1} - \mu_j}{\lambda_{k-1}} \|v_j\|_{X_0}^2 + \frac{\mu_j - \lambda_{k+m}}{\lambda_{k+m}} \|w_j\|_{X_0}^2 \\ &\leq -\frac{\bar{\sigma}}{\lambda_{k-1}} \|v_j\|_{X_0}^2 - \frac{\bar{\sigma}}{\lambda_{k+m}} \|w_j\|_{X_0}^2 \\ &\leq -\frac{\bar{\sigma}}{\lambda_{k+m}} \|u_j\|_{X_0}^2, \end{aligned}$$

thanks again to the properties of the projections v_j and w_j of u_j , respectively on \mathbb{H}_{k-1} and on \mathbb{P}_{k+m-1} , and to (5.1).

On the other hand, by the Hölder inequality, (1.8) and the fact that $X_0 \hookrightarrow L^q(\Omega)$ compactly, there exists a suitable positive constant $\tilde{\kappa}$, independent of j , such that

$$\begin{aligned} \left| \int_{\Omega} f(x, u_j(x))(v_j(x) - w_j(x)) dx \right| &\leq \|f(\cdot, u_j(\cdot))\|_{L^{q/(q-1)}(\Omega)} \|v_j - w_j\|_{L^q(\Omega)} \\ &\leq \tilde{\kappa} \|f(\cdot, u_j(\cdot))\|_{L^{q/(q-1)}(\Omega)} \|v_j - w_j\|_{X_0} \\ &= \tilde{\kappa} \|f(\cdot, u_j(\cdot))\|_{L^{q/(q-1)}(\Omega)} \|u_j\|_{X_0}, \end{aligned}$$

so that, by this and (5.7), we deduce that

$$\tilde{\kappa} \|f(\cdot, u_j(\cdot))\|_{L^{q/(q-1)}(\Omega)} \|u_j\|_{X_0} \geq \frac{\bar{\sigma}}{\lambda_{k+m}} \|u_j\|_{X_0}^2.$$

Being $u_j \not\equiv 0$ by assumption (see (5.2)), we get

$$(5.8) \quad \|f(\cdot, u_j(\cdot))\|_{L^{q/(q-1)}(\Omega)} \geq \frac{\bar{\sigma}}{\tilde{\kappa} \lambda_{k+m}} \|u_j\|_{X_0}$$

for any $j \in \mathbb{N}$.

With the previous estimates, we are now ready to show that

$$(5.9) \quad \text{the sequence } \{\|u_j\|_{X_0}\}_{j \in \mathbb{N}} \text{ is bounded in } \mathbb{R}.$$

For this it is enough to use (1.8) and (1.9), which yield for any $j \in \mathbb{N}$

$$\begin{aligned}
 \int_{\Omega} |f(x, u_j(x))|^{q/(q-1)} dx &\leq \int_{\Omega} (a_1 + a_2 |u_j(x)|^{q-1})^{q/(q-1)} \\
 (5.10) \qquad \qquad \qquad &\leq \tilde{a}_1 + \tilde{a}_2 \int_{\Omega} |u_j(x)|^q dx \\
 &\leq \tilde{a}_3 + \tilde{a}_4 \int_{\Omega} F(x, u_j(x)) dx,
 \end{aligned}$$

for suitable positive constants \tilde{a}_i , $i = 1, \dots, 4$. By (5.5), (5.8) and (5.10) we get assertion (5.9).

In view of (5.9) and (5.2), we can assume that there exists $u_{\infty} \in \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}$ such that

$$\begin{aligned}
 (5.11) \quad &\int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y))(\varphi(x) - \varphi(y)) K(x - y) dx dy \rightarrow \\
 &\int_{\mathbb{R}^n \times \mathbb{R}^n} (u_{\infty}(x) - u_{\infty}(y))(\varphi(x) - \varphi(y)) K(x - y) dx dy \quad \text{for any } \varphi \in X_0,
 \end{aligned}$$

while, by [32, Lemma 8] and [5, Theorem IV.9], up to a subsequence,

$$\begin{aligned}
 (5.12) \quad &u_j \rightarrow u_{\infty} \quad \text{in } L^q(\mathbb{R}^n) \\
 &u_j \rightarrow u_{\infty} \quad \text{a.e. in } \mathbb{R}^n
 \end{aligned}$$

as $j \rightarrow +\infty$ and there exists $\ell \in L^q(\mathbb{R}^n)$ such that

$$(5.13) \quad |u_j(x)| \leq \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}.$$

Moreover, by (2.3) and (5.8) we get that for any $\varepsilon > 0$ there exists C_{ε} such that

$$\begin{aligned}
 (5.14) \quad 0 &< \frac{\bar{\sigma}}{\tilde{\kappa} \lambda_{k+m}} \leq \frac{\|f(\cdot, u_j(\cdot))\|_{L^{q/(q-1)}(\Omega)}}{\|u_j\|_{X_0}} \\
 &\leq \frac{\left(\int_{\Omega} (2\varepsilon |u_j(x)| + qC_{\varepsilon} |u_j(x)|^{q-1})^{q/(q-1)} dx \right)^{(q-1)/q}}{\|u_j\|_{X_0}} \\
 &\leq \frac{\left(2^{1/(q-1)} \left((2\varepsilon)^{q/(q-1)} \|u_j\|_{L^{q/(q-1)}(\Omega)}^{q/(q-1)} + (qC_{\varepsilon})^{q/(q-1)} \|u_j\|_{L^q(\Omega)}^q \right) \right)^{(q-1)/q}}{\|u_j\|_{X_0}} \\
 &\leq \frac{2\varepsilon \|u_j\|_{L^{q/(q-1)}(\Omega)} + qC_{\varepsilon} \|u_j\|_{L^q(\Omega)}^{q-1}}{\|u_j\|_{X_0}} \\
 &\leq C \left(2\varepsilon + qC_{\varepsilon} \|u_j\|_{X_0}^{q-2} \right),
 \end{aligned}$$

thanks to the continuous embedding $X_0 \hookrightarrow L^{\nu}(\Omega)$ for any $\nu \in [1, 2^*)$, and for some universal positive constant C .

By (1.8), (1.12), (5.12), (5.13) and the Dominated Convergence Theorem, it is easily seen that

$$(5.15) \quad \int_{\Omega} F(x, u_j(x)) dx \rightarrow \int_{\Omega} F(x, u_{\infty}(x)) dx$$

and

$$(5.16) \quad \int_{\Omega} |f(x, u_j(x))|^{q/(q-1)} dx \rightarrow \int_{\Omega} |f(x, u_{\infty}(x))|^{q/(q-1)} dx$$

as $j \rightarrow +\infty$. Relation (5.15), combined with (1.11), the fact that $F(x, 0) = 0$ a.e. $x \in \Omega$, and (5.5), yields that

$$(5.17) \quad u_{\infty} \equiv 0.$$

Now, two cases can occur. If

$$(5.18) \quad u_j \rightarrow u_\infty \equiv 0 \quad \text{strongly in } X_0$$

as $j \rightarrow +\infty$, then, by (5.14) we get that

$$0 < \frac{\bar{\sigma}}{\tilde{\kappa}\lambda_{k+m}} \leq 2C\varepsilon,$$

which gives a contradiction, due to the fact that ε is arbitrary. Otherwise, there exists $\eta > 0$ such that $\|u_j\|_{X_0} \geq \eta$ for j large enough. Then, by this, (5.8), (5.16), (5.17) and the fact that $f(x, 0) = 0$ a.e. $x \in \Omega$ (by (1.10)), we get that

$$\frac{\bar{\sigma}\eta}{\tilde{\kappa}\lambda_{k+m}} \leq 0,$$

which is a contradiction. This completes the proof of Lemma 6. \square

The second lemma we need in order to prove the ∇ -condition is the following one:

Lemma 7. *Let k and m in \mathbb{N} be such that $\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$, let $\lambda \in \mathbb{R}$ and f be a function satisfying conditions (1.7)–(1.11). Moreover, let $\{u_j\}_{j \in \mathbb{N}}$ be a sequence in X_0 such that*

$$(5.19) \quad \{\mathcal{J}_\lambda(u_j)\}_{j \in \mathbb{N}} \text{ is bounded in } \mathbb{R},$$

$$(5.20) \quad P_{\text{span}\{e_k, \dots, e_{k+m-1}\}} u_j \rightarrow 0 \text{ in } X_0$$

and

$$(5.21) \quad P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \nabla \mathcal{J}_\lambda(u_j) \rightarrow 0 \text{ in } X_0$$

as $j \rightarrow +\infty$.

Then, $\{u_j\}_{j \in \mathbb{N}}$ is bounded in X_0 .

Proof. Assume by contradiction that $\{u_j\}_{j \in \mathbb{N}}$ is unbounded in X_0 ; without loss of generality, we can assume that

$$(5.22) \quad \|u_j\|_{X_0} \rightarrow +\infty$$

as $j \rightarrow +\infty$ and that there exists $u_\infty \in X_0$ such that

$$(5.23) \quad \begin{aligned} \frac{u_j}{\|u_j\|_{X_0}} &\rightharpoonup u_\infty \text{ in } X_0 \\ \frac{u_j}{\|u_j\|_{X_0}} &\rightarrow u_\infty \text{ in } L^\nu(\Omega) \text{ for any } \nu \in [1, 2^*) \end{aligned}$$

as $j \rightarrow +\infty$ and for any $\nu \in [1, 2^*)$ there exists $\ell_\nu \in L^\nu(\mathbb{R}^n)$ such that

$$(5.24) \quad |u_j(x)| \leq \ell_\nu(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}.$$

Now, for simplicity, we set $P_{\text{span}\{e_k, \dots, e_{k+m-1}\}} =: P$ and $P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} =: Q$, and write

$$u_j = Pu_j + Qu_j,$$

where $Pu_j \rightarrow 0$ as $j \rightarrow \infty$ (see (5.20)). Recalling (2.9) and (2.10), we have

$$(5.25) \quad \begin{aligned} \langle Q \nabla \mathcal{J}_\lambda(u_j), u_j \rangle_{X_0} &= \langle \nabla \mathcal{J}_\lambda(u_j), u_j \rangle_{X_0} - \langle P \nabla \mathcal{J}_\lambda(u_j), u_j \rangle_{X_0} \\ &= \|u_j\|_{X_0}^2 - \lambda \int_\Omega |u_j(x)|^2 dx - \int_\Omega f(x, u_j(x)) u_j(x) dx \\ &\quad - \langle P(u_j - \mathcal{L}_K^{-1}(\lambda u_j + f(x, u_j))), u_j \rangle_{X_0}. \end{aligned}$$

Since $\langle Pu, v \rangle_{X_0} = \langle u, Pv \rangle_{X_0}$ for any $u, v \in X_0$, we have

$$(5.26) \quad \begin{aligned} \langle P(u_j - \mathcal{L}_K^{-1}(\lambda u_j + f(x, u_j))), u_j \rangle_{X_0} &= \|Pu_j\|_{X_0}^2 - \lambda \langle Pu_j, \mathcal{L}_K^{-1} u_j \rangle_{X_0} \\ &\quad - \langle Pu_j, \mathcal{L}_K^{-1} f(x, u_j) \rangle_{X_0}, \end{aligned}$$

while, by (2.11), we obtain

$$(5.27) \quad \begin{aligned} \lambda \langle Pu_j, \mathcal{L}_K^{-1} u_j \rangle_{X_0} + \langle Pu_j, \mathcal{L}_K^{-1} f(x, u_j) \rangle_{X_0} &= \lambda \int_{\Omega} |Pu_j(x)|^2 dx \\ &+ \int_{\Omega} f(x, u_j(x)) Pu_j(x) dx. \end{aligned}$$

Therefore, by (5.25)-(5.27), we get

$$(5.28) \quad \begin{aligned} \langle Q \nabla \mathcal{J}_{\lambda}(u_j), u_j \rangle_{X_0} &= 2 \mathcal{J}_{\lambda}(u_j) + 2 \int_{\Omega} F(x, u_j(x)) dx - \int_{\Omega} f(x, u_j(x)) u_j(x) dx \\ &- \|Pu_j\|_{X_0}^2 + \lambda \int_{\Omega} |Pu_j(x)|^2 dx + \int_{\Omega} f(x, u_j(x)) Pu_j(x) dx. \end{aligned}$$

By (5.19)–(5.22) and (5.28) we easily get that

$$(5.29) \quad \frac{2 \int_{\Omega} F(x, u_j(x)) dx - \int_{\Omega} f(x, u_j(x)) u_j(x) dx + \int_{\Omega} f(x, u_j(x)) Pu_j(x) dx}{\|u_j\|_{X_0}^q} \rightarrow 0$$

as $j \rightarrow +\infty$.

Now, let us show that

$$(5.30) \quad u_{\infty} \equiv 0.$$

For this purpose, we firstly claim that

$$(5.31) \quad \frac{\int_{\Omega} f(x, u_j(x)) Pu_j(x) dx}{\|u_j\|_{X_0}^q} \rightarrow 0$$

as $j \rightarrow +\infty$. Indeed, by (1.8) and (5.24), we have that a.e. $x \in \Omega$

$$\begin{aligned} |f(x, u_j(x)) Pu_j(x)| &\leq \|Pu_j\|_{\infty} (a_1 + a_2 |u_j(x)|^{q-1}) \\ &\leq \|Pu_j\|_{\infty} (a_1 + a_2 |\ell_q(x)|^{q-1}), \end{aligned}$$

while, by (5.20) and the fact that all norms in \mathbb{H}_{k+m-1} are equivalent,

$$\|Pu_j\|_{\infty} \rightarrow 0$$

as $j \rightarrow +\infty$. Note that $Pu_j \in L^{\infty}(\Omega)$, since all eigenfunctions of \mathcal{L}_K are bounded (see [28, Proposition 2.4]). Hence, (5.31) holds.

By this and (5.29) we obtain that

$$0 \leftarrow \frac{2 \int_{\Omega} F(x, u_j(x)) dx - \int_{\Omega} f(x, u_j(x)) u_j(x) dx}{\|u_j\|_{X_0}^q} \leq \frac{(2-q) \int_{\Omega} F(x, u_j(x)) dx}{\|u_j\|_{X_0}^q} \leq 0,$$

as $j \rightarrow +\infty$, also thanks to (1.11). Hence,

$$\frac{\int_{\Omega} F(x, u_j(x)) dx}{\|u_j\|_{X_0}^q} \rightarrow 0$$

as $j \rightarrow +\infty$.

As a consequence of this, (1.9) and (5.22) we have that

$$\frac{\int_{\Omega} |u_j(x)|^q dx}{\|u_j\|_{X_0}^q} \rightarrow 0,$$

as $j \rightarrow +\infty$, which yields (5.30), thanks to (5.23).

Now, by (5.19) and (5.22), we get

$$\frac{\mathcal{J}_\lambda(u_j)}{\|u_j\|_{X_0}^2} = \frac{1}{2} - \frac{\lambda}{2} \frac{\int_{\Omega} |u_j(x)|^2 dx}{\|u_j\|_{X_0}^2} - \frac{\int_{\Omega} F(x, u_j(x)) dx}{\|u_j\|_{X_0}^2} \rightarrow 0,$$

which, together with (5.23) (here with $\nu = 2$) and (5.30), implies that

$$(5.32) \quad \frac{\int_{\Omega} F(x, u_j(x)) dx}{\|u_j\|_{X_0}^2} \rightarrow \frac{1}{2}$$

as $j \rightarrow +\infty$.

Hence, as a consequence of (1.9), (5.22) and (5.32), there exists $C > 0$ such that

$$(5.33) \quad \|u_j\|_{L^q(\Omega)}^q \leq C \|u_j\|_{X_0}^2 \text{ for every } j \in \mathbb{N}.$$

Now, let us show that

$$(5.34) \quad \frac{\int_{\Omega} f(x, u_j(x)) P u_j(x) dx}{\|u_j\|_{X_0}^2} \rightarrow 0$$

as $j \rightarrow +\infty$. Indeed, by (1.8) and the Hölder inequality, we have

$$\begin{aligned} \frac{\int_{\Omega} |f(x, u_j(x)) P u_j(x)| dx}{\|u_j\|_{X_0}^2} &\leq \frac{\|P u_j\|_{\infty}}{\|u_j\|_{X_0}^2} \left(a_1 |\Omega| + a_2 \int_{\Omega} |u_j(x)|^{q-1} dx \right) \\ &\leq \|P u_j\|_{\infty} \left[\frac{a_1 |\Omega|}{\|u_j\|_{X_0}^2} + \frac{a'_2}{\|u_j\|_{X_0}^{2/q}} \left(\frac{\int_{\Omega} |u_j(x)|^q dx}{\|u_j\|_{X_0}^2} \right)^{1-1/q} \right] \\ &\leq \|P u_j\|_{\infty} \left[\frac{a_1 |\Omega|}{\|u_j\|_{X_0}^2} + \frac{a'_2 C^{1-1/q}}{\|u_j\|_{X_0}^{2/q}} \right], \end{aligned}$$

thanks to (5.33). Thus, (5.34) follows from this, (5.20) and (5.22).

Finally, dividing both sides of (5.28) by $\|u_j\|_{X_0}^2$, using (5.19)–(5.22) and (5.34) we get

$$\frac{2 \int_{\Omega} F(x, u_j(x)) dx - \int_{\Omega} f(x, u_j(x)) u_j(x) dx}{\|u_j\|_{X_0}^2} \rightarrow 0,$$

which, arguing as above, yields

$$\frac{2 \int_{\Omega} F(x, u_j(x)) dx}{\|u_j\|_{X_0}^2} \rightarrow 0,$$

as $j \rightarrow +\infty$. Of course, this is in contradiction with (5.32). The proof of Lemma 7 is complete. \square

As a consequence of Lemma 6 and Lemma 7, we get the following result on the validity of the ∇ -condition for \mathcal{J}_λ .

Proposition 8. *Let k and m in \mathbb{N} be such that $\lambda_{k-1} < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$ and let f be a function satisfying conditions (1.7)–(1.11).*

Then, for any $\sigma > 0$ there exists $\varepsilon_\sigma > 0$ such that for any $\lambda \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ and for any $\varepsilon', \varepsilon'' \in (0, \varepsilon_\sigma)$, with $\varepsilon' < \varepsilon''$, functional \mathcal{J}_λ satisfies the $(\nabla)(\mathcal{J}_\lambda, \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}, \varepsilon', \varepsilon'')$ condition.

Proof. Assume by contradiction that there exists $\sigma > 0$ such that for every $\varepsilon_0 > 0$ there exist $\bar{\lambda} \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ and $\varepsilon' < \varepsilon''$ in $(0, \varepsilon_0)$ such that

$$(5.35) \quad (\nabla)(\mathcal{J}_{\bar{\lambda}}, \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}, \varepsilon', \varepsilon'') \text{ does not hold.}$$

Associated to such a σ , take $\varepsilon_0 > 0$ as provided by Lemma 6.

By (5.35) we can find a sequence $\{u_j\}_{j \in \mathbb{N}}$ in X_0 such that

$$(5.36) \quad \begin{aligned} \mathcal{J}_{\bar{\lambda}}(u_j) &\in [\varepsilon', \varepsilon''] \text{ for all } j \in \mathbb{N}, \\ \text{dist}(u_j, \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}) &\rightarrow 0 \\ P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \nabla \mathcal{J}_{\bar{\lambda}}(u_j) &\rightarrow 0 \text{ in } X_0 \end{aligned}$$

as $j \rightarrow +\infty$.

By Lemma 7 we know that $\{u_j\}_{j \in \mathbb{N}}$ is bounded in X_0 , and so we can assume that for some $u_\infty \in X_0$

$$(5.37) \quad \begin{aligned} u_j &\rightharpoonup u_\infty \text{ in } X_0 \\ u_j &\rightarrow u_\infty \text{ in } L^\nu(\Omega) \text{ for any } \nu \in [1, 2^*) \\ u_j &\rightarrow u_\infty \text{ a.e. in } \Omega \end{aligned}$$

as $j \rightarrow +\infty$ and for any $\nu \in [1, 2^*)$ there exists $\ell_\nu \in L^\nu(\mathbb{R}^n)$ such that

$$(5.38) \quad |u_j(x)| \leq \ell_\nu(x) \text{ a.e. in } \mathbb{R}^n \text{ for any } j \in \mathbb{N}.$$

Now, note that by (2.10) we can write

$$(5.39) \quad \begin{aligned} P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \nabla \mathcal{J}_{\bar{\lambda}}(u_j) &= u_j - P_{\text{span}\{e_k, \dots, e_{k+m-1}\}} u_j \\ &\quad - P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \mathcal{L}_K^{-1}(\bar{\lambda} u_j + f(x, u_j)). \end{aligned}$$

Hence, recalling that $\mathcal{L}_K^{-1} : L^{q'}(\Omega) \rightarrow X_0$ is a compact operator (see Section 2.4), and that $f(x, u_j) \rightarrow f(x, u_\infty)$ in $L^{q'}(\Omega)$ by (1.7)-(1.8) and (5.38), we get that

$$P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \mathcal{L}_K^{-1}(\bar{\lambda} u_j + f(x, u_j)) \rightarrow P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \mathcal{L}_K^{-1}(\bar{\lambda} u_\infty + f(x, u_\infty))$$

as $j \rightarrow +\infty$ and so, taking into account (5.36), (5.37) and (5.39), we deduce that

$$(5.40) \quad u_j \rightarrow P_{\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}} \mathcal{L}_K^{-1}(\bar{\lambda} u_\infty + f(x, u_\infty)) =: u_\infty \text{ in } X_0$$

as $j \rightarrow +\infty$.

Furthermore, again by (5.36) we have that

$$\langle \nabla \mathcal{J}_{\bar{\lambda}}(u_j), \varphi \rangle_{X_0} \rightarrow 0 \text{ for any } \varphi \in \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1},$$

that is, taking into account (2.9),

$$(5.41) \quad \langle \mathcal{J}'_{\bar{\lambda}}(u_j), \varphi \rangle = \langle u_j, \varphi \rangle_{X_0} - \lambda \int_{\Omega} u_j(x) \varphi(x) dx - \int_{\Omega} f(x, u_j(x)) \varphi(x) dx \rightarrow 0$$

as $j \rightarrow +\infty$. Thus, by (1.8), (5.37), (5.40) and (5.41), we obtain that

$$\langle \mathcal{J}'_{\bar{\lambda}}(u_\infty), \varphi \rangle = \langle u_\infty, \varphi \rangle_{X_0} - \lambda \int_{\Omega} u_\infty(x) \varphi(x) dx - \int_{\Omega} f(x, u_\infty(x)) \varphi(x) dx$$

for any $\varphi \in \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}$, i.e. u_∞ is a critical point of $\mathcal{J}_{\bar{\lambda}}$ constrained on $\mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}$.

Hence, Lemma 6 yields that $u_\infty \equiv 0$. However, $0 < \varepsilon' \leq \mathcal{J}_{\bar{\lambda}}(u_j)$ for every $j \in \mathbb{N}$, so that, by continuity of $\mathcal{J}_{\bar{\lambda}}$, we find $\mathcal{J}_{\bar{\lambda}}(u_\infty) > 0$, which is absurd. This completes the proof of Proposition 8. \square

6. PROOF OF MAIN THEOREM

This section is devoted to the proof of main result of the paper, concerning the existence of multiple solutions for problem (1.2). In order to get this result we apply the following abstract critical point theorem ([14, Theorem 2.10]):

Theorem 9 (Sphere-torus linking with mixed type assumptions). *Let H be a Hilbert space and X_1, X_2, X_3 be three subspaces of H such that $H = X_1 \oplus X_2 \oplus X_3$ with $0 < \dim X_i < \infty$ for $i = 1, 2$. Let $\mathcal{I} : H \rightarrow \mathbb{R}$ be a $C^{1,1}$ functional. Let $\rho, \rho', \rho'', \rho_1$ be such that $0 < \rho_1$, $0 \leq \rho' < \rho < \rho''$ and*

$$\Delta = \{u \in X_1 \oplus X_2 : \rho' \leq \|P_2 u\| \leq \rho'', \|P_1 u\| \leq \rho_1\} \text{ and } T = \partial_{X_1 \oplus X_2} \Delta,$$

where $P_i : H \rightarrow X_i$ is the orthogonal projection of H onto X_i , $i = 1, 2$, and

$$S_{23}(\rho) = \{u \in X_2 \oplus X_3 : \|u\| = \rho\} \text{ and } B_{23}(\rho) = \{u \in X_2 \oplus X_3 : \|u\| < \rho\}.$$

Assume that

$$a' = \sup \mathcal{I}(T) < \inf \mathcal{I}(S_{23}(\rho)) = a''.$$

Let a, b be such that $a' < a < a''$, $b > \sup \mathcal{I}(\Delta)$ and

the assumption $(\nabla)(\mathcal{I}, X_1 \oplus X_3, a, b)$ holds;

the Palais–Smale condition holds at any level $c \in [a, b]$.

Then, \mathcal{I} has at least two critical points in $\mathcal{I}^{-1}([a, b])$.

If, furthermore,

$$a_1 < \inf \mathcal{I}(B_{23}(\rho)) > -\infty$$

and the Palais–Smale condition holds at every $c \in [a_1, b]$, then \mathcal{I} has another critical level between a_1 and a' .

First of all, we need the following result:

Lemma 10. *Let k and m in \mathbb{N} be such that $\lambda_{k-1} < \lambda < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$ and let f satisfy (1.7)–(1.11).*

Then, the following relation is verified:

$$\lim_{\lambda \rightarrow \lambda_k} \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_\lambda(u) = 0.$$

Proof. First of all, note that \mathcal{J}_λ attains a maximum in \mathbb{H}_{k+m-1} by (1.9).

Now, assume by contradiction that there exist $\{\mu_j\}_{j \in \mathbb{N}}$, such that

$$(6.1) \quad \mu_j \rightarrow \lambda_k$$

as $j \rightarrow +\infty$, $\{u_j\}_{j \in \mathbb{N}}$ in \mathbb{H}_{k+m-1} and $\varepsilon > 0$ such that for any $j \in \mathbb{N}$

$$(6.2) \quad \mathcal{J}_{\mu_j}(u_j) = \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_{\mu_j}(u) \geq \varepsilon.$$

If $\{u_j\}_{j \in \mathbb{N}}$ is bounded in X_0 , we can assume that $u_j \rightarrow u_\infty$ in X_0 as $j \rightarrow +\infty$ for some $u_\infty \in \mathbb{H}_{k+m-1}$. Then, by (2.4), (6.1) and the fact that \mathbb{H}_{k+m-1} is finite-dimensional, we have that

$$\mathcal{J}_{\mu_j}(u_j) \rightarrow \mathcal{J}_{\lambda_k}(u_\infty)$$

as $j \rightarrow +\infty$, and so by (1.11), (2.8) and (6.2), we immediately get

$$\begin{aligned} \varepsilon &\leq \mathcal{J}_{\lambda_k}(u_\infty) = \frac{1}{2} \|u_\infty\|_{X_0}^2 - \frac{\lambda_k}{2} \int_{\Omega} |u_\infty(x)|^2 dx - \int_{\Omega} F(x, u_\infty(x)) dx \\ &\leq \frac{1}{2} (\lambda_{k+m-1} - \lambda_k) \int_{\Omega} |u_\infty(x)|^2 dx - \int_{\Omega} F(x, u_\infty(x)) dx \leq 0, \end{aligned}$$

and a contradiction arises.

Otherwise, if $\{u_j\}_{j \in \mathbb{N}}$ is unbounded in X_0 , we can suppose that

$$(6.3) \quad \|u_j\|_{X_0} \rightarrow +\infty$$

as $j \rightarrow +\infty$. Therefore, (6.2) and (1.9) imply

$$0 < \varepsilon \leq \mathcal{J}_{\mu_j}(u_j) \leq \frac{1}{2} \|u_j\|_{X_0}^2 - \frac{\mu_j}{2} \int_{\Omega} |u_j(x)|^2 dx - a_3 \int_{\Omega} |u_j(x)|^q dx + a_4 |\Omega|.$$

But all norms are equivalent in \mathbb{H}_{k+m-1} , so that the right hand side of the previous inequality tends to $-\infty$ as $j \rightarrow +\infty$, since $q > 2$ by assumption, and (6.3) holds true. Hence, a contradiction arises as well. \square

Now, we can prove our multiplicity result for problem (1.2). The idea consists in applying Theorem 9 to \mathcal{J}_{λ} , in connection with a classical Linking Theorem (see [27, Theorem 5.3]). First we prove:

Proposition 11. *Let k and m in \mathbb{N} be such that $\lambda_{k-1} < \lambda < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}$ and let f satisfy (1.7)–(1.11).*

Then, there exists a left neighborhood \mathcal{O}_k of λ_k such that for all $\lambda \in \mathcal{O}_k$, problem (1.2) has two nontrivial solutions u_i such that

$$0 < \mathcal{J}_{\lambda}(u_i) \leq \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_{\lambda}(u)$$

for $i = 1, 2$.

Proof. The strategy consists in applying Theorem 9 to the functional \mathcal{J}_{λ} . For this purpose, fix $\sigma > 0$ and find ε_{σ} as in Proposition 8. Then, for all $\lambda \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ and for every $\varepsilon', \varepsilon'' \in (0, \varepsilon_{\sigma})$, functional \mathcal{J}_{λ} satisfies the $(\nabla)(\mathcal{J}_{\lambda}, \mathbb{H}_{k-1} \oplus \mathbb{P}_{k+m-1}, \varepsilon', \varepsilon'')$ condition.

By Lemma 10 there exists $\sigma_1 \leq \sigma$ such that, if $\lambda \in (\lambda_k - \sigma_1, \lambda_k)$, then

$$(6.4) \quad \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_{\lambda}(u) < \varepsilon''.$$

Moreover, since $\lambda < \lambda_k$, Proposition 5 holds true. Also, \mathcal{J}_{λ} satisfies the Palais–Smale condition at any level, by Proposition 4.

Then, by Theorem 9, there exist two critical points u_1, u_2 of \mathcal{J}_{λ} with

$$(6.5) \quad \mathcal{J}_{\lambda}(u_i) \in [\varepsilon', \varepsilon''],$$

$i = 1, 2$. In particular u_1 and u_2 are non-trivial solutions of problem (1.2) such that

$$0 < \mathcal{J}_{\lambda}(u_i) \leq \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_{\lambda}(u) \quad i = 1, 2,$$

since ε'' is arbitrary in (6.4) and (6.5), and this ends the proof of Proposition 11. \square

We are now ready to conclude with the

Proof of Theorem 2. By the classical Linking Theorem (see [27, Theorem 5.3]), for any $\lambda \in (\lambda_{k-1}, \lambda_k)$ one can prove the existence of a solution u_3 of problem (1.2) with

$$(6.6) \quad \mathcal{J}_{\lambda}(u_3) \geq \inf_{u \in \mathbb{P}_{k-1}, \|u\|=\varrho} \mathcal{J}_{\lambda}(u) \geq \beta,$$

for suitable $\varrho > 0$ and $\beta > 0$, see [33].

By Lemma 10, we can choose λ so close to λ_k that

$$(6.7) \quad \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_{\lambda}(u) < \inf_{u \in \mathbb{P}_{k-1}, \|u\|=\varrho} \mathcal{J}_{\lambda}(u).$$

Hence, inequalities (6.6), (6.7) and Proposition 11 immediately imply that

$$\mathcal{J}_{\lambda}(u_i) \leq \sup_{u \in \mathbb{H}_{k+m-1}} \mathcal{J}_{\lambda}(u) < \mathcal{J}_{\lambda}(u_3)$$

and so $u_3 \neq u_i$, $i = 1, 2$. The proof of Theorem 2 is complete. \square

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